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ESTIMATING THE DISTRIBUTION OF ADDITIVE REACTION-TIME COMPONENTS[1]

by

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One of the oldest ideas in experimental psychology is that the time between stimulus and response is occupied by a series of stages, with one stage being initiated upon the completion of the preceding one. At the very least, stages for perception, decision, and response have been proposed. This serial theory has led to numerous models that represent the reaction time as a sum of random variables. It has usually been assumed that the component random variables are independent. Moreover, most of these models go further, and specify some particular form for the component distributions.

One of the objects of the work I shall report on today is to test the independence of reaction-time components without having to assume any particular form for their distributions. Support for the independence assumption would also be evidence favoring a more general theory of serial processing. My second object is to estimate the unknown distributions of the components.

The problem of decomposing an observed reaction-time distribution into a sum of independent components has been discussed before, notably in 1956 by Christie and Luce[2] and, more recently, by McGill.[3] But in both cases the proposed methods require the form of at least one of the component distributions to be specified.

Christie and Luce did, however, define a class of experimental situations to which the testing and estimation methods I shall describe are applicable. In such situations the reaction time includes a series of identically distributed components whose number is under experimental control. Thus, in my experiments, the subject first memorizes a small set of digits. On each trial a test digit is presented. If the test digit is one of those in memory the subject pulls one lever, making a positive or "yes" response. If the test digit is not one of those in memory the subject makes a negative response by pulling the other lever. The time from the onset of the test digit to the response is measured. Results of a number of experiments suggest that in deciding whether or not a test digit is contained in memory, the subject searches exhaustively in the memorized set at a rate of about 30 symbols per second. That is, he matches or compares the test digit successively with each of the memorized digits, with a mean comparison time of about 35 milliseconds. The size of the memorized set, and therefore the number of successive comparisons, is controlled by the experimenter.

Some of the experiments giving support to this theory were described at last year's meeting of the Psychonomic Society.[4]

The model that I shall consider is represented by Equation 1.

$$RT(s) = T_b + T_1 + T_2 + \ldots + T_s, \text{ where } \begin{cases} T_b = \text{base time} \\ T_i \ (i = 1,\ldots,s) = \text{comparison time.} \end{cases} \qquad (1)$$

If s elements are in memory, then the reaction-time is given by the sum of s "comparison" times, denoted by T_i, and a "base" time, denoted by T_b. The base time is a wastebasket category, and presumably represents perception and response times, as well as some of the decision time. In general, positive and negative responses have different base times. In a typical experiment, three conditions are studied, in which the values of s are one, two and four. For both positive and negative responses, one observes distributions of the three sums given in (2).

$$T_b + T_1, \quad T_b + T_1 + T_2, \quad T_b + T_1 + T_2 + T_3 + T_4 \qquad (2)$$

I shall describe a procedure for testing, jointly, two assumptions about the component random variables. The first is that they are independent. The second is that the comparison times, T_i, are identically distributed. Consequences of these assumptions are best described in terms of the cumulants, κ_r, of the reaction-time distribution. You will probably recall that when the cumulants exist, they are generated by the natural logarithm of the moment generating function, as shown in Equation 3.

$$\mathrm{KGF} = \kappa_1\theta + \kappa_2 \frac{\theta^2}{2!} + \kappa_3 \frac{\theta^3}{3!} + \ldots$$

$$= \log_e\left(1 + \alpha_1'\theta + \alpha_2' \frac{\theta^2}{2!} + \ldots\right) = \log_e \mathrm{MGF}. \qquad (3)$$

In (4) the first four cumulants are given in terms of the first four moments. α_1' is the first raw moment; α_r is the rth central moment.

$$\kappa_1 = \alpha_1', \ \kappa_2 = \alpha_2, \ \kappa_3 = \alpha_3, \ \kappa_4 = \alpha_4 - 3\alpha_2^2. \qquad (4)$$

The symbols to be used are defined as follows:

	Cumulants	Moments
Reaction-time distribution	κ_r	
Base-time distribution	β_r	ν_r
Comparison-time distribution	γ_r	μ_r
Contaminated RT distribution	C_r	

In general, the rth cumulant can be expressed in terms of moments of order r and lower. Because MGF's of independent random variables multiply, KGF's add. For all r, therefore, the rth cumulant of a sum of independent random variables is the sum of the rth cumulants of the individual variables.

Equation 5 follows from this additivity property.

$$\kappa_r(s) = \gamma_r s + \beta_r, \quad (r = 1,2,\ldots). \tag{5}$$

If the model is correct, then for all r, the rth cumulant is a linear function of s. The slope of this function is the rth cumulant of the comparison-time distribution; its intercept is the corresponding cumulant of the base-time distribution.

Before considering applications of Equation 5, let us turn briefly to the problem of contamination of the reaction-time distribution. There are not many students of reaction time who feel that their experimental technique is sufficiently clean so that they can completely eliminate uninteresting contamination from their data. No matter what precautions are taken, response movements are occasionally abortive, or the subject may blink just when the stimulus is presented. If the amount of contamination depends on the value of s, then we are in serious difficulty. A plausible model for contamination that is <u>not</u> dependent on s is given below.

$$RT(s) = \begin{cases} T_b + T_1 + \ldots + T_s, & \text{with prob } 1-\varepsilon, \\ T_b + T_1 + \ldots + T_s + X, & \text{with prob } \varepsilon. \end{cases} \tag{6}$$

With a small probability, ε, a random variable, X, is added to the normal reaction time. Equation 7 shows the effect of such contamination on the cumulant functions.

$$C_r(s) = \gamma_r s + (\beta_r + \varepsilon B_r), \qquad (r = 1,2,\ldots). \tag{7}$$

B_r is a function of ε and the cumulants of X. The effect on the slope is nil, and the effect on the intercept is roughly proportional to ε. Most important, the functions remain linear.

The cumulants of a distribution are estimated without bias by k-statistics, which are functions of the sample moments.[5] Because sampling error increases rapidly with r, especially in the presence of contamination, I decided to restrict my attention to cumulants of orders one to four. These are capable of providing a good deal of interesting information. If the model is correct, each of these cumulants should increase linearly with s.

In addition, the model requires certain relations to hold among the slopes and among the intercepts of the four linear functions. If you refer to Equation 5 again you will see that the four slopes provide estimates of the first four cumulants of the comparison-time distribution, and the four intercepts provide estimates of the corresponding cumulants of the base time distribution. We therefore have estimates of the first four moments of each of these hypothetical distributions. If they are in fact moments of a non-degenerate probability distribution, it can be shown[6] that they must satisfy inequalities 8.

$$\alpha_2 > 0, \quad \alpha_4 > \alpha_2^2 + \frac{\alpha_3^2}{\alpha_2} . \tag{8}$$

Moreover, since these are hypothetical distributions of <u>times</u>, which cannot be negative, each set of moment estimates must also satisfy inequalities 9.

$$\alpha_1' > 0, \quad \alpha_3 > \frac{\alpha_2^2}{\alpha_1'} - \alpha_2\alpha_1' . \tag{9}$$

Suppose that the model is supported by these tests. In that case one is justified in extracting further information from the moment estimates. First, given a particular family of distributions, one can ask whether the estimated moments can have arisen from one of its members. One family, for example, that has been popular among model builders, contains the exponential distribution and convolutions of various numbers of exponentials with equal or unequal time constants. If the hypothetical comparison-time distribution is one of these, for example, then its moments must satisfy inequalities 11a and 11b.[7]

$$0 < \alpha_3 \leq 2\alpha_2^{3/2} . \tag{11a}$$

$$3\alpha_2^2 + \frac{3\alpha_3^2}{2\alpha_2} \leq \alpha_4 \leq 3\alpha_2^2 + 2.381\ \alpha_3^{4/3} . \tag{11b}$$

The second use to which the moment estimates can be put is in estimating the underlying distributions themselves. One solution to this problem is provided by the Pearson system of frequency curves, and the associated method of moments. Pearson's system was originally conceived in order to fit curves to <u>observed</u> frequency distributions. But it was later used in theoretical statistics to approximate <u>unknown</u> distributions whose lower moments could be calculated. The four-parameter Pearson system contains many of the well-known families of distributions, including the normal, beta, exponential and gamma, t, F and Pareto distributions. Estimates of the first four moments can be used to determine both the family to which the hypothetical distribution belongs, and its parameter values. And a knowledge of the distribution of comparison times, for example, may contribute to our understanding of the unobservable comparison process.

I have applied these testing and estimation procedures to the data from three experiments. The experiments had been designed for other purposes, so the results of this application should not be taken too seriously. On the whole, they appear promising. I analyzed only the times for negative responses, that is, responses made when the test digit is <u>not</u> one of those in memory. For each subject in an experiment, and for values of s of 1,2, and 4, the first four k-statistics were determined. Their values were then averaged over subjects. Figure 1 (page 10) shows the resulting mean value of each of the first four k-statistics, as a function of s. The curves labeled A and C each represent data from twelve subjects; curves labeled B represent data from six subjects. The functions appear to deviate systematically from linearity. It is a serious weakness of the method that one cannot determine which assumption is at fault - independence, identity of distributions, or invariance of contamination effects with the number of elements in memory. Among other things, an experiment with a larger range of s-values is called for. Also, it would be pleasant to have cumulant estimates that are less sensitive to extreme observations. In the data presented here, the evidence most favorable to the model is provided by Experiment A.

Despite the apparent nonlinearity of some of the mean k-statistics, I proceeded to estimate the moments of the hypothetical comparison- and base-time distributions. Assuming that the underlying functions were linear, I used a simple least squares procedure to estimate slopes and intercepts. The resulting moment estimates are plotted in Figure 2 (page 10) There is good agreement among the estimates from the three

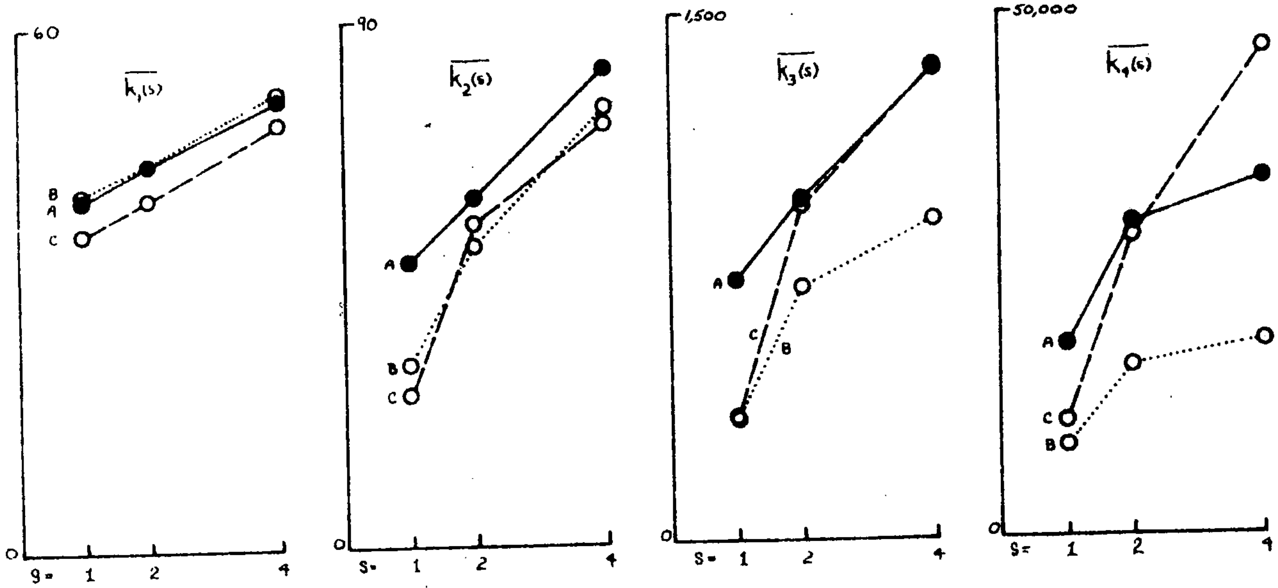

Figure 1
Mean values of the first four k-statistics versus s
in experiments A, B and C.
Time unit = .01 sec.
(Note that ordinate scales differ.)

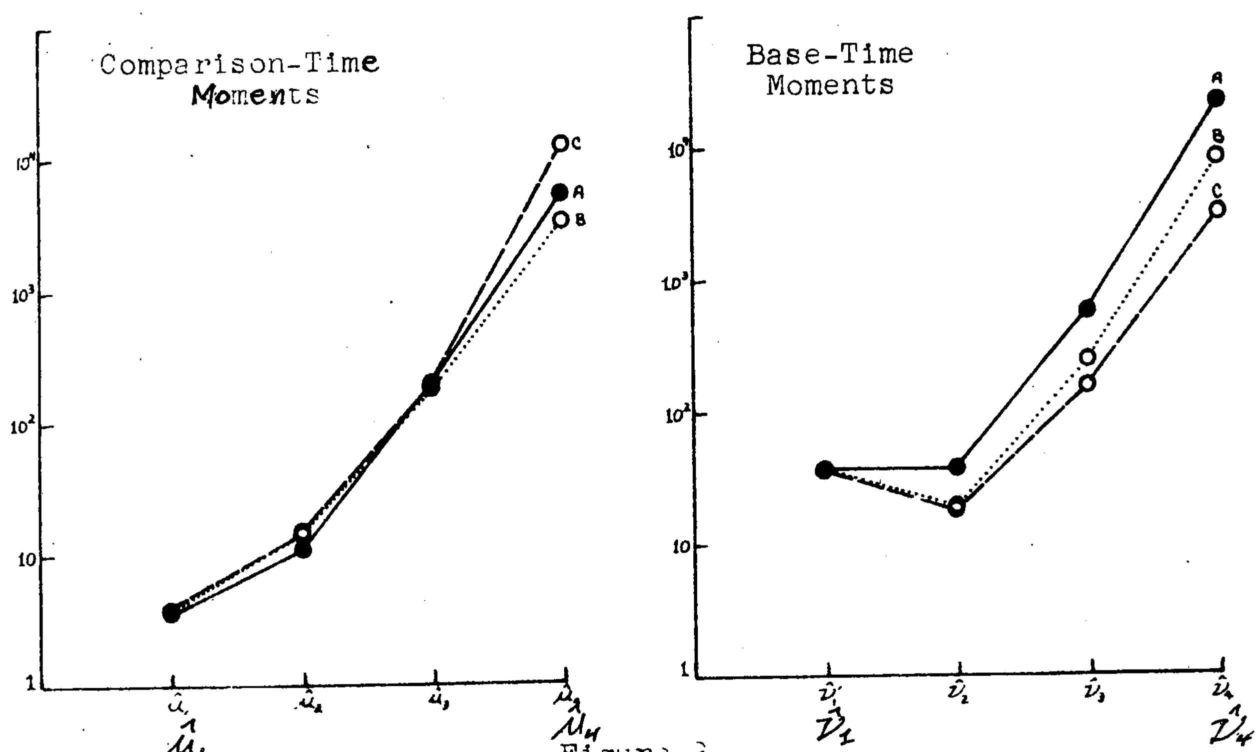

Figure 2
Estimated values of four moments
of comparison-time and base-time distributions
in experiments A, B and C.
Time unit = .01 sec.
(Log scale)

experiments, at least for the comparison-time distribution. From these figures emerges the fact that whereas a change in s affects the mean reaction time by only a small percentage, it has a relatively large effect on the higher moments.

Six sets of moments are displayed in Figure 2. If they in fact represent moments of time distributions, then they ought to satisfy inequalities 9 and 10. On testing the estimated moments I found that the inequalities were satisfied by each of the six sets.[8]

These results encouraged me to proceed further. By applying inequalities 11a and 11b, I found that none of these sets of moments could have arisen from an exponential distribution, or from a convolution of any number of exponentials. Both the third and fourth moments are too large relative to the second. This result is especially interesting for Experiment A. Here, as in many reaction-time experiments, the observed distributions can be fitted reasonably well by gamma distributions. We thus have an illustration of the fact that even if a gamma distribution appears to arise from a sum of independent components, these components need not be exponentials or gammas.

Because the data from Experiment A seemed to fit the model best, I entered the Pearson system with the moments estimated from that experiment. The results are shown in Figures 3 and 4 (page 12). The estimated distribution of the

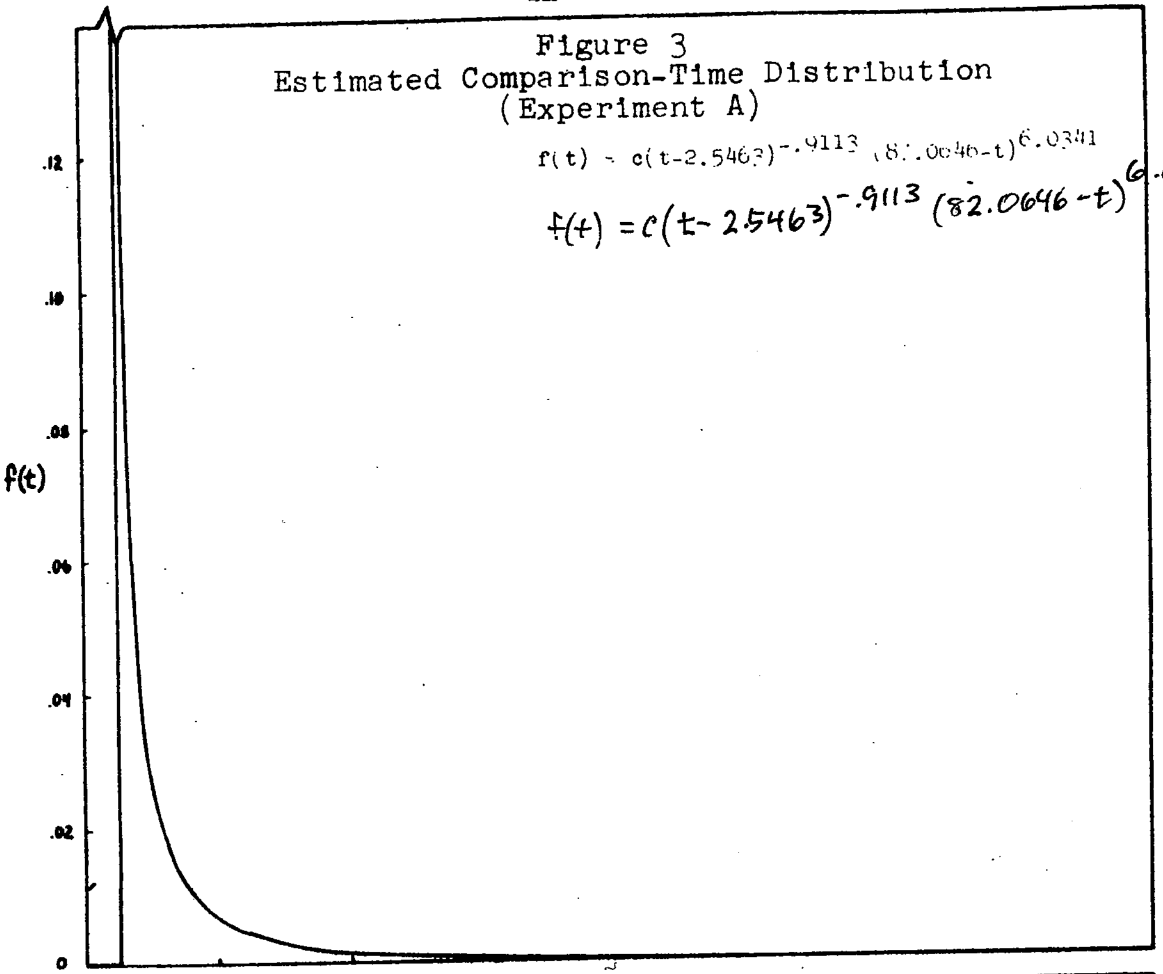

Figure 3
Estimated Comparison-Time Distribution
(Experiment A)

$$f(t) = c(t-2.5463)^{-.9113}(82.0646-t)^{6.0341}$$

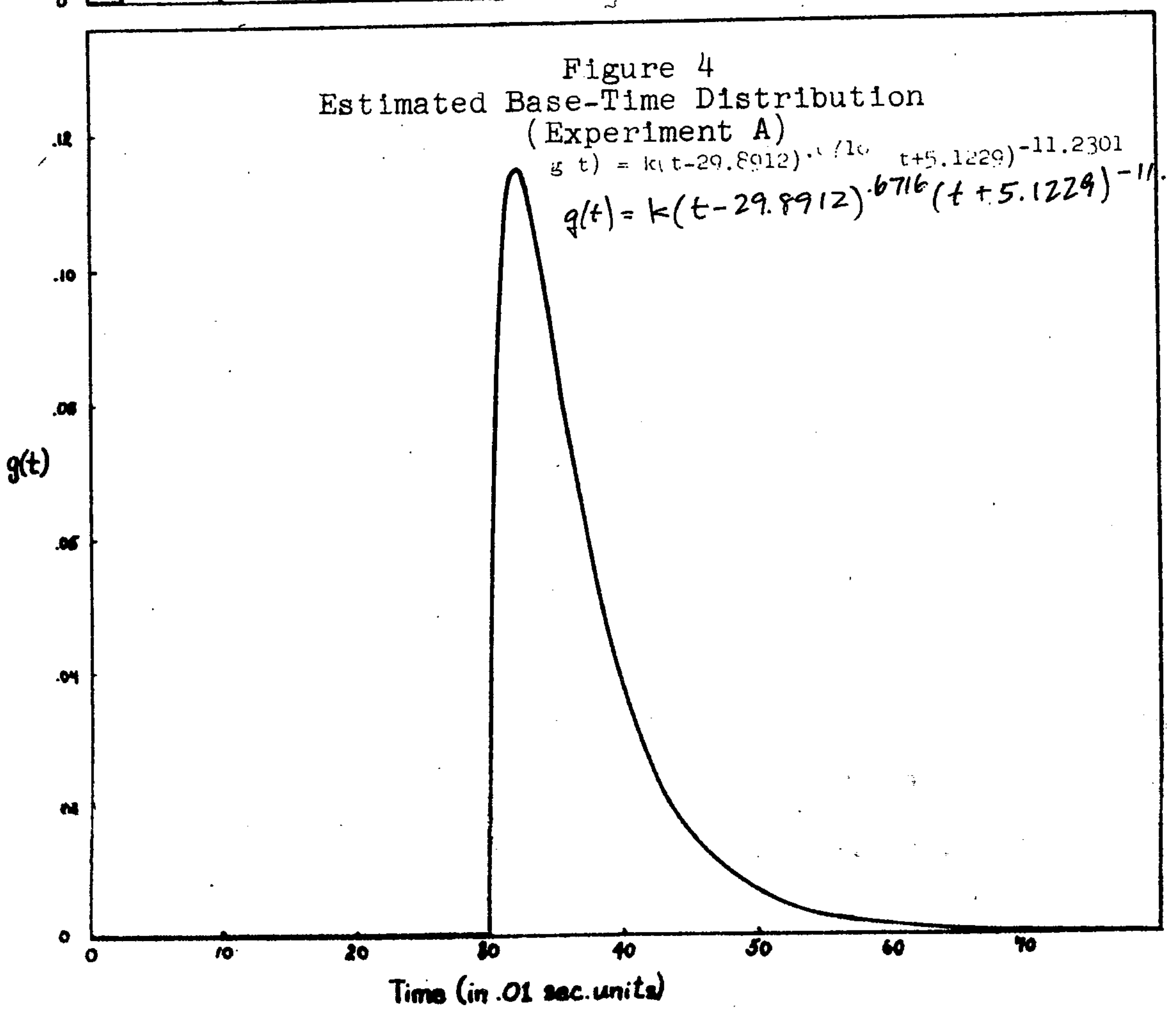

Figure 4
Estimated Base-Time Distribution
(Experiment A)

$$g(t) = k(t-29.8912)^{.6716}(t+5.1229)^{-11.2301}$$

comparison time is a Pearson Type I curve. This is essentially a beta distribution. In units of .01 second, its range starts at about 2.5, where the probability density is infinite, and its mean is about 3.6. The estimated distribution of the base time is a Pearson Type VI curve. This is a generalization of the F distribution. In units of .01 second, its range starts at about 30. Its mode is about 32 and its mean is about 36.

In conclusion I would like to underscore two important limitations of these methods. The first is that they are applicable only when there is reason to believe that the experimenter can control the number of reaction-time components, without also affecting the base time.[9] The second is that with these methods, the assumption of independence of components cannot be tested in isolation, but only in conjunction with a second assumption, that the components are identically distributed.[10]

NOTES AND REFERENCES

[1]The experiments on which this study is based were performed at the University of Pennsylvania with the support of Grant GB-1172 from the National Science Foundation. I am indebted to C. L. Mallows and R. Gnanadesikan of the Bell Telephone Laboratories for helpful discussions.

[2]Christie, L. S. and Luce, R. D., Decision structure and time relations in simple choice behavior. Bull. Math Biophysics, 1956, 18, 89-112.

[3]McGill, W. J. and Gibbon, J., The general-gamma distribution and reaction times. Tech. Rep. No. 61, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1963.

[4]Sternberg, S., Retrieval from recent memory: Some reaction-time experiments and a search theory. Psychonomic Society, August 1963.

[5]Let $m_r' = \frac{1}{n} \sum_{j=1}^{n} x_j^r$ and $m_r = \frac{1}{n} \sum_{j=1}^{n} (x_j - m_1')^r$. Then the first four k-statistics of a sample $x_1, x_2, \ldots, x_n$ are given in terms of these moment statistics as follows:

$$k_1 = m_1', \quad k_2 = \frac{n}{n-1} m_1, \quad k_3 = \frac{n^2}{(n-1)(n-2)} m_3,$$

$$k_4 = \frac{n^2(n+1)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n^2}{(n-2)(n-3)} m_2^2 .$$

See Kendall, M. G. and Stuart, A., The Advanced Theory of Statistics, Vol. 1.

[6]For a summary of these and other moment properties, see Mallows, C. L., Note on the moment problem for unimodal distributions when one or both terminals are known, Biometrika, 1956, 43, 224-227.

[7]See Appendix for derivation.

[8]Each of the six sets of moments also satisfied the necessary and sufficient conditions that a distribution be unimodal. (Johnson, N L. and Rogers, C. A., The moment problem for unimodal distributions, Ann. Math. Stat., 1951, 22, 433-439.)

[9]If one wishes simply to test the assumption of independence of several components this requirement can be weakened. Consider the case of two components. Suppose that we have an experimental situation in which it is thought that the reaction time includes at least two distinct serial components, T_1 and T_2, so that $RT = T_b + T_1 + T_2$. T_1 and T_2 might, for example, be times for detection and for motor response. Suppose, further, that we have an operation, O_1, that alters T_1 without affecting T_2, and another operation, O_2, that alters T_2 without affecting T_1. (Neither of these operations affects the number of components.) Then we can create four experimental situations:

$$
\begin{array}{ll}
(\overline{O}_1,\overline{O}_2): & RT_{00} = T_b + T_1 + T_2 \\
(O_1,\overline{O}_2): & RT_{10} = T_b + T_1' + T_2 \\
(\overline{O}_1,O_2): & RT_{01} = T_b + T_1 + T_2' \\
(O_1,O_2): & RT_{11} = T_b + T_1' + T_2'
\end{array}
$$

Let $\kappa_r(i)$ be the rth cumulant of T_i, and $\kappa_r(i) + \Delta\kappa_r(i)$ be the rth cumulant of T_i'. Also, let $K_r(\ \ )$ be the rth cumulant of the observed reaction times in the situation indicated in parentheses. If T_1 and T_2 are independent as well as being additive, then <u>cumulant changes must be additive</u>, as follows:

$$K_r(10) - K_r(00) = \Delta\kappa_r(1)$$

$$K_r(01) - K_r(00) = \Delta\kappa_r(2)$$

and

$$K_r(11) - K_r(00) = \Delta\kappa_r(1) + \Delta\kappa_r(2),$$

for $r = 1,2,\ldots$. These equations imply that, for all r,

$$K_r(11) = K_r(01) + K_r(10) - K_r(00)$$

Assuming only additivity, this relation must hold for the means ($r = 1$); this fact has led to its use as a test of additivity of reaction-time components (e.g., McMahon, L. E., Grammatical Analysis as Part of Understanding a Sentence, Unpublished Ph.D. dissertation, Harvard University, 1963).

[10] If independence is <u>assumed</u>, then only two adjacent values of s are needed in order to use the estimation method, and the assumption that components are identically distributed is not required. Under these circumstances the estimation procedure is a strong version of the Helmholtz-Donders subtraction method.

APPENDIX

Loci of Sums of Independent Exponential Random Variables in the Pearson (β_1,β_2)-Plane

Let $T = T_1 + T_2 \dots + T_k$, where the T_i are independent and exponentially distributed with density functions

$$f_i(t_i) = \frac{1}{\lambda_i} e^{-t_i/\lambda_i}, \quad (\lambda_i > 0,\ t_i \geq 0).$$

The MGF of T_i is $(1-\lambda_i\theta)^{-1}$ and the KGF is therefore

$$-\log(1-\lambda_i\theta) = \sum_{r=1}^{\infty} \frac{(\lambda_i\theta)^r}{r} = \sum_{r=1}^{\infty} (r-1)!\ \lambda_i^r \frac{\theta^r}{r!},$$

which shows the rth cumulant to be $\gamma_r(i) = (r-1)!\ \lambda_i^r$. By the additivity of cumulants,

$$\kappa_r = \sum_{i=1}^{k} \gamma_r(i) = (r-1)! \sum_{i=1}^{k} \lambda_i^r,$$

where κ_r is the rth cumulant of T. Thus

$$\mu_1' = \kappa_1 = \Sigma\lambda_i,$$

$$\mu_2 = \kappa_2 = \Sigma\lambda_i^2,$$

$$\mu_3 = \kappa_3 = 2\Sigma\lambda_i^3,$$

$$\mu_4 - 3\mu_2^2 = \kappa_4 = 6\Sigma\lambda_i^4,$$

and

$$\beta_1 \equiv \frac{\mu_3^2}{\mu_2^3} = 4\,\frac{\left(\Sigma\lambda_i^3\right)^2}{\left(\Sigma\lambda_i^2\right)^3}\,,$$

$$\beta_2 \equiv \frac{\mu_4}{\mu_2^2} = 3 + \frac{\kappa_4}{\kappa_2^2} = 3 + 6\,\frac{\Sigma\lambda_i^4}{\left(\Sigma\lambda_i^2\right)^2}\,.$$

Equal time constants. If $\lambda_i = \lambda (i = 1,2,\ldots,k)$ then the betas become

$$\beta_1 = \frac{4}{k}\,, \qquad \beta_2 = 3 + \frac{6}{k}\,.$$

These define a set of points (for $k = 1,2,\ldots$) in the (β_1,β_2)-plane that fall on the line $\beta_2 = 3 + \frac{3}{2}\beta_1$. Corresponding to the entire line ($\beta_1 > 0$) is the family of gamma distributions. Corresponding to the points on the line produced by integral k,

$$(\beta_1,\beta_2) = (4,9),\ (2,6),\ (1\tfrac{1}{3},5),\ (1,4\tfrac{1}{2}),\ \ldots$$

are those gamma distributions (special Erlangian distributions) produced by convolutions of exponentials with equal time constants. (The limit, (0,3), of this sequence of points corresponds to the normal distribution.)

Unequal time constants. (An exhaustive review of what is known about this situation is provided by McGill and Gibbon.[3]) Generous bounds on the region occupied by (β_1,β_2) are easily provided.

Consider β_1 first. Its lower bound is zero. To determine its upper bound, one can use the fact that for $0 < r < s$,

$$\left(\Sigma\lambda_i^s\right)^r < \left(\Sigma\lambda_i^r\right)^s$$

(from Theorem 19 - Jensen's Inequality - in Hardy, G. H., Littlewood, J. E. and Polya, G., Inequalities. Cambridge University Press, 1959.) This shows that the quantity $\beta_1/4$ is bounded above by unity. Thus $0 < \beta < 4$, which is equivalent to inequality 11a. To discover bounds on β_2 one can form the following quotient:

$$\frac{4^2}{6^3} \frac{(\beta_2-3)^3}{\beta_1^2} = \frac{\left(\Sigma\lambda_i^4\right)^3}{\left(\Sigma\lambda_i^3\right)^4} .$$

Again by Jensen's Inequality, the right-hand member is bounded above by unity. This gives $\beta_2 \leq 3 + (27/2)^{\frac{1}{3}} \beta_1^{\frac{2}{3}}$. Differentiation shows that its minimum is attained when all the λ_i are equal, in which case its value is $1/k$, and (β_1,β_2) is one of the points corresponding to equal time constants, so that $\beta_2 \geq 3 + \frac{3}{2}\beta_1$. The result is that β_2 must be contained in the narrow region between $3 + \frac{3}{2}\beta_1$ and $3 + (27/2)^{\frac{1}{3}} \beta_1^{\frac{2}{3}}$, with $0 < \beta_1 \leq 4$. An equivalent statement is provided by 11b. These bounds can be improved upon.

Because central moments are unaffected by a translation of the density function, the results also hold when the components are exponentials with "dead time," that is, when

$$f_i(t_i) = \frac{1}{\lambda_i} e^{-(t-\tau_i)/\lambda_i}, \quad (\lambda_i > 0,\ t \geq \tau_i).$$