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Limit Theorems for Additive Learning Models[1]

M. FRANK NORMAN

University of Pennsylvania, Philadelphia, Pennsylvania 19104

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In the learning models considered, the events that occur on successive trials act additively. New derivations are given of a criterion for recurrence or transience of response probability, and a formula for the asymptotic response frequency in the recurrent case. It is shown that there are stationary probability distributions in the recurrent case, and that they are approximately normal with small variance when learning occurs by small steps.

1. INTRODUCTION

Consider a situation in which on each of a sequence of trials a subject makes one of two choices, A_0 or A_1. Response A_i has j_i possible consequences O_{ij}, $j = 1,..., j_i$, and the probability π_{ij} of O_{ij} given A_i is fixed throughout the experiment. An interesting class of learning models is characterized by the assumption that the effects of the outcomes O_{ij} are additive at the level of some real response strength variable L. Let L_n and p_n be a subject's A_1 response strength and A_1 response probability on the n-th trial, and let $A_{i,n}$ and $O_{ij,n}$ be the events response A_i and outcome O_{ij} on trial n. Such a model specifies a strictly increasing function $p(L)$ from R (the real numbers) onto I (the open unit interval) such that $p_n = p(L_n)$, and $\Delta L_n = L_{n+1} - L_n = b_{ij}$ if $A_{i,n}$ and $O_{ij,n}$, for some constant b_{ij}. Clearly these transformations of L (and consequently the induced transformations of p) are commutative. The beta model (Luce, 1959) is of this type [$p = v/(v + 1)$ where $v = e^L$], as are all models generated by quasi-additive families of operators (Marley, 1967) on p.

This paper considers several aspects of the asymptotic behavior of stochastic processes $\{L_n\}$ and $\{p_n\}$ defined in Sec. 2 that are slightly more general than those discussed above. Lamperti and Suppes (1960) showed that, for the beta model, questions concerning recurrence and absorption hinge on the signs of the means $m_i = \sum_j b_{ij}\pi_{ij}$, and Luce (1964) and Marley (1967) noted that the form of the function p was not crucial to Lamperti's and Suppes' arguments. In Sec. 3 bounds are obtained for the sequence $E[v_n^\lambda]$ for certain real λ, and these are used to give a new derivation of the recurrence-absorption classification based on m_i. All of the remaining sections are

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concerned with the recurrent case $m_0 > 0$, $m_1 < 0$, and make use of the corresponding bounds on $E[v_n^\lambda]$. Let $X_n = 1$ if $A_{1,n}$ and $X_n = 0$ if $A_{0,n}$. In Sec. 4 a new proof is given of a result due to Holman (1969, Theorem 4.1), to the effect that the proportion $(1/n) \sum_{j=1}^{n} X_j$ of A_1 responses in the first n trials converges to $\rho = m_0/(m_0 - m_1)$ almost surely (a.s.) as $n \to \infty$. The limit of $(1/n) \sum_{j=1}^{n} E[v_j^k]$ for the beta model is also obtained. It is shown in Sec. 5 that the Markov process $\{L_n\}$ possesses at least one stationary probability distribution, and problems relating to its uniqueness are discussed. Finally, in Sec. 6 it is shown that, if all b_{ij} are proportional to a parameter θ, then the stationary probability distributions of $\{p_n\}$ are asymptotically normal with mean ρ and variance proportional to θ as $\theta \to 0$.

In Secs. 3 and 4 substantial use is made of martingale theory (Neveu, 1965, Secs. IV.5 and IV.6), the relevance of which to problems in this area was clearly established by Lamperti (1960, 1963).

2. Assumptions

In the last section the conditional distribution Q_i of ΔL_n given $A_{i,n}$ was concentrated at a finite number of points b_{ij}, $j = 1,..., j_i$. This discreteness would play no role in most of our work, and it is, moreover, easy to conceive of two choice experiments where the responses have continua of possible consequences, so we shall not assume it in general. Rather we will assume only that Q_i possesses a moment generating function. Similarly, the hypothesis of Sec. 1 that p is a strictly increasing mapping of R onto I is much stronger than necessary for most of Secs. 3-5, so we will impose a somewhat weaker condition. Our assumption is equivalent to the condition that, for any sequence L_n, $p(L_n) \to 1$ if and only if $L_n \to \infty$, and $p(L_n) \to 0$ if and only if $L_n \to -\infty$. Under this condition p may be constant or may even decrease over a finite interval. The latter possibility has no obvious psychological interpretation.

Here, then, are our general assumptions.

A. Q_i, $i = 0, 1$, is a probability distribution over the Borel subsets of R, such that $M_i(\lambda) = \int_{-\infty}^{\infty} e^{\lambda x} Q_i(dx)$ exists for λ in some open interval J containing 0.

B. p is a measurable mapping of R into I, such that $p(L) \to 1$ as $L \to \infty$, $p(L) \to 0$ as $L \to -\infty$, and p is bounded away from 0 and 1 on any finite interval. Let $q = 1 - p$.

C. $T_n = (L_n, X_n)$, $n \geq 1$, is a bivariate stochastic process in $R \times \{0, 1\}$, such that

$$P(X_n = 1 \mid L_n, T_{n-1}, ..., T_1) = p_n (p_n = p(L_n)),$$
$$P(X_n = 0 \mid L_n, T_{n-1}, ..., T_1) = q_n (q_n = q(L_n)),$$

and

$$P(\Delta L_n \in A \mid T_n, ..., T_1) = Q_{X_n}(A)$$

almost surely.

It is clear that L_n is a Markov process with stationary transition probabilities, hence $v_n = e^{L_n}$ is too. In Sections 3-5 it will always be assumed that the distribution of L_1 is concentrated at a point L. When we wish to emphasize the initial state, we write $P_L(\cdot)$ or $E_L[\cdot]$ instead of simply $P(\cdot)$ or $E[\cdot]$. The implications of our results for arbitrary initial distributions are easily obtained by conditioning on L_1.

The importance of the $M_i(\lambda)$ is suggested by the equation

$$E[(v_{n+1}/v_n)^\lambda \mid v_n] = M_1(\lambda)\, p_n + M_0(\lambda)\, q_n \tag{1}$$

a.s., which shows that these quantities determine the tendency of $v_n{}^\lambda$ to rise or fall. It follows from A that, for all $k \geqslant 0$ and $\lambda \in J$, $\int_{-\infty}^{\infty} x^k e^{\lambda x} Q_i(dx)$ and $M_k^{(i)}(\lambda)$ exist and are equal. In particular, if $m_i = \int_{-\infty}^{\infty} x Q_i(dx)$, then $m_i = M_i'(0)$. Thus m_i determines the departure of $M_i(\lambda)$ from $M_i(0) = 1$ when $|\lambda|$ is small. If $m_i < 0$ then $M_i(\lambda) < 1$ for $\lambda > 0$ sufficiently small, while if $m_i > 0$ then $M_i(\lambda) < 1$ for $\lambda < 0$ sufficiently large. Actually, since $M_i''(\lambda) = \int_{-\infty}^{\infty} x^2 e^{\lambda x} Q_i(dx) \geqslant 0$, M_i is convex, so that $M_i(\lambda) < 1$ implies that $M_i(\omega) < 1$ whenever ω is between λ and 0.

Throughout the rest of the paper it will be understood that an argument of M_i is an element of J.

3. Recurrence

By assumption B, when L_n is very large, p_n is near 1, so $A_{1,n}$ is very likely. Lemmas 1a and 2a show that $+\infty$ acts as a reflecting or absorbing barrier for L_n, depending on whether m_1, the conditional expectation of ΔL_n given $A_{1,n}$, is negative or positive. Lemmas 1b and 2b give comparable results for $-\infty$.

For two events A and B we say that A implies B a.s. if $P(A - B) = 0$. We write $\lim L_n$ instead of $\lim_{n\to\infty} L_n$.

Lemma 1. a. *If* $m_1 < 0$, $\liminf L_n < \infty$ a.s. *For any* $\lambda > 0$ *such that* $M_1(\lambda) < 1$, *there is a constant* $B(\lambda)$ *such that* $\limsup E_L[v_n{}^\lambda] \leqslant B(\lambda)$.

b. *If* $m_0 > 0$, $\limsup L_n > -\infty$ a.s. *For any* $\lambda < 0$ *such that* $M_0(\lambda) < 1$, *there is a constant* $B(\lambda)$ *such that* $\limsup E_L[v_n{}^\lambda] \leqslant B(\lambda)$.

Lemma 2. a. *If* $m_1 > 0$, *then* $\limsup L_n = \infty$ *implies* $\lim L_n = \infty$ a.s., *and* $g_0(L) = P_L(\lim L_n = -\infty) \to 0$ *as* $L \to \infty$.

b. *If* $m_0 < 0$, *then* $\liminf L_n = -\infty$ *implies* $\lim L_n = -\infty$ a.s., *and* $g_1(L) = P_L(\lim L_n = \infty) \to 0$ *as* $L \to -\infty$.

Proof of Lemma 1. As a consequence of (1), we have

$$E[v_{n+1}^\lambda \mid v_n] = v_n{}^\lambda[M_1(\lambda)\, p_n + M_0(\lambda)\, q_n] \tag{2}$$

a.s. Suppose that $m_1 < 0$, $\lambda > 0$ and $M_1(\lambda) < 1$. Let $M_1(\lambda) < D < 1$. $F_\lambda(v) = M_1(\lambda)\, p(\ln v) + M_0(\lambda)\, q(\ln v) \to M_1(\lambda)$ as $v \to \infty$, hence there is a K such that $F_\lambda(v) \leqslant D$ whenever $v \geqslant K$. Hence, (2) shows that $v_n \geqslant K$ implies $E[v_{n+1}^\lambda \mid v_n] \leqslant D v_n{}^\lambda$ a.s. But, clearly, $v_n \leqslant K$ implies $E[v_{n+1}^\lambda \mid v_n] \leqslant K^\lambda N(\lambda)$ a.s., where $N(\lambda) = \max(M_1(\lambda), M_0(\lambda))$. Therefore, $E[v_{n+1}^\lambda \mid v_n] \leqslant D v_n{}^\lambda + K^\lambda N(\lambda)$ a.s. Taking expectations on both sides, we obtain $E[v_{n+1}^\lambda] \leqslant D E[v_n{}^\lambda] + K^\lambda N(\lambda)$. Iteration then yields

$$E[v_n{}^\lambda] \leqslant v_1{}^\lambda D^{n-1} + K^\lambda N(\lambda)(1 - D^{n-1})/(1 - D).$$

Therefore, $\limsup E[v_n{}^\lambda] \leqslant B(\lambda)$, where $B(\lambda) = K^\lambda N(\lambda)/(1 - D)$. Since

$$\liminf E[v_n{}^\lambda] < \infty,$$

Fatou's lemma implies that $E[\liminf v_n{}^\lambda] < \infty$. Therefore, $\liminf v_n{}^\lambda < \infty$ a.s. and $\liminf L_n < \infty$ a.s. This completes the proof of *a*. The proof of *b* is similar.

Q.E.D.

Proof of Lemma 2. Suppose $m_1 > 0$. Choose $\lambda < 0$ such that $M_1(\lambda) < 1$. Choose C such that $v_n \geqslant C$ implies $E[v_{n+1}^\lambda \mid v_n] \leqslant v_n{}^\lambda$ a.s. Let $H(v, \lambda) = \min(v^\lambda, C^\lambda)$ and $H_n = H(v_n, \lambda)$. $\{H_n\}$ is a supermartingale. For

$$E[H_{n+1} \mid v_n, \ldots, v_1] \leqslant E[v_{n+1}^\lambda \mid v_n] \leqslant v_n{}^\lambda = H_n$$

a.s. if $v_n \geqslant C$, while

$$E[H_{n+1} \mid v_n, \ldots, v_1] \leqslant E[C^\lambda \mid v_n] = C^\lambda = H_n$$

a.s. for $v_n \leqslant C$. Hence $E[H_{n+1} \mid v_n, \ldots, v_1] \leqslant H_n$ a.s., as claimed. Since $H_n \geqslant 0$, $\lim H_n$ exists a.s. (Neveu, 1965, (1) on p. 137). If $\limsup L_n = \infty$, $\liminf H_n = 0$. But the latter implies $\lim H_n = 0$ a.s., which in turn implies $\lim L_n = \infty$. It follows that $\limsup L_n = \infty$ implies $\lim L_n = \infty$ a.s.

To complete the proof of *a*, we note that $E[H_n] \leqslant H_1$ for all n, so $E[\lim H_n] \leqslant H_1$. But $\lim H_n \geqslant C^\lambda I_A$ where A is the event $\lim L_n = -\infty$ and $I_A = 1$ or 0 depending on whether or not A occurs. Thus $P(A) \leqslant H_1/C^\lambda$. For $v \geqslant C$ this becomes $g_0(L) \leqslant e^{\lambda L}/C^\lambda$, and the quantity on the right converges to 0 as $L \to \infty$. Q.E.D.

The next lemma is of a rather different kind.

Lemma 3. a. *If* $m_0 > 0$ *or* $m_1 > 0$, $P(\limsup L_n \in R) = 0$.

b. *If* $m_0 < 0$ *or* $m_1 < 0$, $P(\liminf L_n \in R) = 0$.

Proof. If $m_i > 0$, then $Q_i([2\epsilon, \infty)) > 0$ for some $\epsilon > 0$. For any x

$$\begin{aligned} P(L_{n+1} &\geqslant x + \epsilon \mid L_n, \ldots, L_1) \\ &= Q_1([x - L_n + \epsilon, \infty))\, p_n + Q_0([x - L_n + \epsilon, \infty)) q_n \\ &\geqslant Q_1([2\epsilon, \infty))\, a_x + Q_0([2\epsilon, \infty))\, b_x = \delta_x \end{aligned}$$

a.s. if $|L_n - x| < \epsilon$, where $a_x = \inf_{|L-x|<\epsilon} p(L)$ and $b_x = \inf_{|L-x|<\epsilon} q(L)$. By assumption B, both a_x and b_x are positive, so $\delta_x > 0$.

It follows that $|L_n - x| < \epsilon$ infinitely often (i.o.) implies

$$\sum_{n=1}^{\infty} P(L_{n+1} \geqslant x + \epsilon \mid L_n, \ldots, L_1) = \infty$$

a.s. The latter event is a.s. equivalent to $L_n \geqslant x + \epsilon$ i.o. (Neveu, 1965, Corollary to Proposition IV.6.3). Thus the probability that $|L_n - x| < \epsilon$ i.o. but not $L_n \geqslant x + \epsilon$ i.o. is 0. However, $|\limsup L_n - x| < \epsilon$ implies the latter event, so

$$P(|\limsup L_n - x| < \epsilon) = 0.$$

Since R is a denumerable union of intervals of the form $(x - \epsilon, x + \epsilon)$, a is proved. Essentially the same argument yields b. Q.E.D.

We are now prepared to distinguish four possibilities for the asymptotic behavior of $\{p_n\}$ when neither m_0 nor m_1 is 0.

THEOREM 1. a. *If* $m_0 > 0$ *and* $m_1 > 0$, $\lim p_n = 1$ a.s.

b. *If* $m_0 < 0$ *and* $m_1 < 0$, $\lim p_n = 0$ a.s.

c. *If* $m_0 < 0$ *and* $m_1 > 0$, $g_0(L) + g_1(L) = 1$, *where* $g_i(L) = P_L(\lim p_n = i)$. *In addition* $g_0(L) > 0$, $g_1(L) > 0$, $g_0(L) \to 1$ *as* $L \to -\infty$ *and* $g_1(L) \to 1$ *as* $L \to \infty$.

d. *If* $m_0 > 0$ *and* $m_1 < 0$, *then* $\limsup p_n = 1$ *and* $\liminf p_n = 0$ a.s.

Proof. Suppose $m_1 > 0$. By Lemma 3a $\limsup L_n = -\infty$ or $\limsup L_n = \infty$ a.s. By Lemma 2a, $\limsup L_n = \infty$ implies $\lim L_n = \infty$ a.s., hence $\limsup L_n = -\infty$ or $\lim L_n = \infty$ a.s. When $m_0 > 0$, $\limsup L_n > -\infty$ a.s. by Lemma 1b, so that, when $m_1 > 0$ and $m_0 > 0$, $\lim L_n = \infty$ a.s. This takes care of a, and b can be handled similarly.

Returning now to the case where our only assumption is $m_1 > 0$, we have $\lim L_n = -\infty$ or $\lim L_n = \infty$ a.s. Thus $g_0(L) + g_1(L) = 1$. It then follows from Lemma 2a that $g_1(L) \to 1$ as $L \to \infty$. In particular, there is a constant x such that $g_1(L) > 0$ if $L \geqslant x$. It is easily shown that, for any $n \geqslant 1$ and $L \in R$, $P_L(\Delta L_j \geqslant \epsilon,\ j = 1, \ldots, n) \geqslant (\delta c_L)^n$, where $c_L = \inf_{x \geqslant L} p(x) > 0$ and $\delta = Q_1([\epsilon, \infty)) > 0$ for $\epsilon > 0$ sufficiently small. Hence $P_L(L_{n+1} \geqslant x) > 0$ if n is sufficiently large that $n\epsilon + L \geqslant x$. But $g_1(L) = E_L[g_1(L_{n+1})]$, so $g_1(L) > 0$ for all $L \in R$. The other conclusions in c follow in the same way from $m_0 < 0$.

Suppose $m_1 < 0$. Then $\liminf L_n = \pm\infty$ a.s. (Lemma 3b) and $\liminf L_n < \infty$ a.s. (Lemma 1a), so $\liminf L_n = -\infty$ a.s. Thus $\liminf p_n = 0$ a.s. Similarly $m_0 > 0$ implies $\limsup p_n = 1$ a.s., and d is proved. Q.E.D.

In the case of a and b of Theorem 1, a simpler approach is available that yields a more refined result. Suppose $m_0 < 0$ and $m_1 < 0$. Then for λ positive but sufficiently small, $N(\lambda) = \max(M_0(\lambda), M_1(\lambda)) < 1$. Equation 2 gives $E[v_{n+1}^\lambda \mid v_n] \leqslant N(\lambda) v_n^\lambda$. Taking expectations on both sides we obtain $E[v_{n+1}^\lambda] \leqslant N(\lambda) E[v_n^\lambda]$. Hence, by iteration, $E[v_n^\lambda] \leqslant N(\lambda)^{n-1} v_1^\lambda$. Therefore

$$E\left[\sum_{n=1}^{\infty} v_n^\lambda\right] = \sum_{n=1}^{\infty} E[v_n^\lambda] \leqslant v_1^\lambda/(1 - N(\lambda)). \tag{3}$$

It follows that $\sum_{n=1}^{\infty} v_n^\lambda < \infty$ and thus $p_n \to 0$ a.s. If $p(L)$ approaches 0 at an exponential rate as $L \to -\infty$ (e.g., in the beta model $p(L) \leqslant e^L$) more can be said.

THEOREM 2. *If* $m_0 < 0$, $m_1 < 0$, *and* $\zeta = \limsup_{L \to -\infty} p^{1/|L|} < 1$, *then* $E_L[T] < \infty$, *where* $T = \sum_{n=1}^{\infty} X_n$ *is the total number of* A_1 *responses.*

Proof. Let $\lambda > 0$ be sufficiently small that $N(\lambda) < 1$ and $\zeta < e^{-\lambda}$. Then, for some K, $p \leqslant K v^\lambda$ for all L. Then (3) implies $E[\sum_{n=1}^{\infty} p_n] < \infty$. But $E[p_n] = E[X_n]$, so $E[T] = E[\sum_{n=1}^{\infty} p_n]$. Q.E.D.

4. Asymptotic Response Frequencies

Theorem 3 gives the asymptotic A_1 response frequency in the recurrent case.

THEOREM 3. If $m_0 > 0$ *and* $m_1 < 1$, $(1/n)\sum_{j=1}^{n} p_j \to \rho$ *and* $(1/n)\sum_{j=1}^{n} X_j \to \rho$ a.s. *as* $n \to \infty$, *where* $\rho = m_0/(m_0 - m_1)$.

Proof. Let $Y_n = \Delta L_n - (m_1 p_n + m_0 q_n)$ and $Z_n = X_n - p_n$. Then

$$E[Y_n \mid L_n, \ldots, L_1] = 0,$$

and $E[Z_n \mid L_n, T_{n-1}, \ldots, T_1] = 0$ a.s., so $\{Y_n\}$ and $\{Z_n\}$ are centered sequences (Neveu, 1965, Definition IV.6.1). If $s_i = \int_{-\infty}^{\infty} x^2 Q_i(dx)$, then

$$E[Y_n^2 \mid L_n] \leqslant E[(\Delta L_n)^2 \mid L_n]$$
$$= s_1 p_n + s_0 q_n \leqslant \max(s_1, s_0),$$

a.s. Thus, $E[Y_n^2] = E[E[Y_n^2 \mid L_n]]$ is bounded over n, as is $E[Z_n^2]$. It follows that $(1/n)\sum_{j=1}^{n} Y_j \to 0$ a.s. and $(1/n)\sum_{j=1}^{n} Z_j \to 0$ a.s. as $n \to \infty$ (Neveu, 1965, (2) of Proposition IV.6.1). Since

$$(1/n)\sum_{j=1}^{n} Z_j = (1/n)\sum_{j=1}^{n} X_j - (1/n)\sum_{j=1}^{n} p_j,$$

the two assertions of Theorem 3 are equivalent.

Now

$$(1/n)\sum_{j=1}^{n} Y_j = L_{n+1}/n - L_1/n + (m_0 - m_1)\Big[(1/n)\sum_{j=1}^{n} p_j - \rho\Big],$$

so it remains only to show that $L_{n+1}/n \to 0$ a.s. as $n \to \infty$.

Since $m_1 < 0$ there exists a $\lambda > 0$ such that $M_1(\lambda) < 1$. By Lemma 1a, $E[v_n{}^\lambda]$ is a bounded sequence. Therefore, $\sum_{n=1}^{\infty} E[v_{n+1}^\lambda]/n^2 < \infty$ so that $v_{n+1}^\lambda/n^2 \to 0$ a.s. Taking the logarithm we obtain $\lambda L_{n+1} - 2\ln n \to -\infty$ a.s., from which it follows that $\limsup L_{n+1}/n \leqslant 0$ a.s. Similarly we deduce $\liminf L_{n+1}/n \geqslant 0$ a.s. from $m_0 > 0$ via Lemma 1b. Thus $\lim L_{n+1}/n = 0$ a.s. Q.E.D.

Luce (1959, Theorem 15) gave a formula for $\lim E[v_n{}^k]$ in terms of the quantities $M_i(j)$ for the beta model, under the assumption that the limit exists. No one has yet succeeded in proving that this convergence occurs in the recurrent case. Theorem 4 shows that $(1/n)\sum_{j=1}^{n} E[v_j{}^k]$ converges, and that the limit is given by Luce's formula.

THEOREM 4. *Suppose that $m_0 > 0$, $m_1 < 0$, and $p = v/(v+1)$. For any positive integer k such that $M_1(k) < 1$*

$$\frac{1}{n}\sum_{j=1}^{n} E_L[v_j{}^k] \to \rho\,\frac{M_0(k) - M_1(k)}{1 - M_1(k)}\prod_{i=1}^{k-1}\frac{M_0(i) - 1}{1 - M_1(i)}$$

as $n \to \infty$. For any negative integer k such that $M_0(k) < 1$,

$$\frac{1}{n}\sum_{j=1}^{n} E_L[v_j{}^k] \to (1-\rho)\,\frac{M_1(k) - M_0(k)}{1 - M_0(k)}\prod_{i=k+1}^{-1}\frac{M_1(i) - 1}{1 - M_0(i)}$$

as $n \to \infty$.

Proof. Taking expectations in (2), subtracting $E[v_n{}^\lambda]$, summing over $1 \leqslant n \leqslant m$, and multiplying by $1/m$ we obtain

$$E[v_{m+1}^\lambda]/m - v_1{}^\lambda/m = (M_1(\lambda) - 1)(1/m)\sum_{n=1}^{m} E[v_n{}^\lambda p_n]$$

$$+ (M_0(\lambda) - 1)(1/m)\sum_{n=1}^{m} E[v_n{}^\lambda q_n].$$

If $\lambda > 0$ and $M_1(\lambda) < 1$ or $\lambda < 0$ and $M_0(\lambda) < 1$, $E[v_{m+1}^\lambda]$ is bounded in m by Lemma 1, so

$$(M_1(\lambda) - 1)(1/n)\sum_{j=1}^{n} E[v_j{}^\lambda p_j] + (M_0(\lambda) - 1)(1/n)\sum_{j=1}^{n} E[v_j{}^\lambda q_j] \to 0$$

as $n \to \infty$.

Since $p = vq$ we may replace $v_j{}^\lambda q_j$ by $v_j^{\lambda-1}p_j$. Using the resulting statement inductively in conjunction with $(1/n)\sum_{j=1}^n E[p_j] \to \rho$, which follows from Theorem 3, we get

$$\frac{1}{n}\sum_{j=1}^{n} E[v_j{}^k p_j] \to \rho \prod_{i=1}^{k} \frac{M_0(i)-1}{1-M_1(i)}$$

as $n \to \infty$, for all $k \geqslant 1$ with $M_1(k) < 1$. The first assertion of the theorem then follows from $v^k = v^k p + v^{k-1}p$. The second assertion is proved in the same way. Q.E.D.

5. Stationary Probability Distributions

In this section we consider the recurrent case under the additional assumption that p is continuous. Theorem 5 shows that the Markov process $\{L_n\}$ possesses a stationary probability distribution, and that all such distributions have moment generating functions. Consequently the moments we will need in the next section exist. Let S be the set of stationary probability distributions.

Theorem 5. *If $m_0 > 0$, $m_1 < 0$, and p is continuous, then S is not empty. Let B be as in Lemma 1. Then $\int_{-\infty}^{\infty} e^{\lambda L}\mu(dL) \leqslant B(\lambda)$ for all $\mu \in S$ if $\lambda > 0$ and $M_1(\lambda) < 1$, or $\lambda < 0$ and $M_0(\lambda) < 1$.*

Proof. For any real number L and Borel set A let

$$K(L, A) = Q_1(A - L)\,p + Q_0(A - L)q.$$

This kernel is a transition probability function for $\{L_n\}$. The operator $T\mu(A) = \int_{-\infty}^{\infty} \mu(dL)\,K(L, A)$ carries the distribution of L_n into the distribution of L_{n+1}.

Let $\lambda > 0$ be sufficiently small that $M_1(\lambda) < 1$ and $M_0(-\lambda) < 1$. Let $L \in R$, let ν be the probability distribution concentrated at L, and let $\nu_n = (1/n)\sum_{j=0}^{n-1} T^j\nu$. Then

$$\int_{-\infty}^{\infty} \cosh \lambda x \nu_n(dx) = (1/n)\sum_{j=1}^{n} E_L[\cosh \lambda L_j] \leqslant C_L$$

for some constant C_L by Lemma 1. Since $\cosh \lambda x$ is an increasing function of $|x|$, $\nu_n(D_y) \leqslant C_L/\cosh \lambda y$ for all $n \geqslant 1$ and $y > 0$ where $D_y = (-\infty, -y) \cup (y, \infty)$. But $\cosh \lambda y \to \infty$ as $y \to \infty$, so $\{\nu_n\}$ is uniformly tight, hence conditionally weakly compact (Parthasarathy, 1967, Theorem II.6.7). Let $\{\nu_{n_j}\}$ be a subsequence that converges weakly to a probability distribution μ as $j \to \infty$.

Since p is continuous, the operator $Uf(x) = \int_{-\infty}^{\infty} f(y)\,K(x, dy)$ maps continuous functions into continuous functions. It then follows from $\int_{-\infty}^{\infty} Uf(x)\,\mu(dx) =$

$\int_{-\infty}^{\infty} f(x)\, T\mu(dx)$ that T is continuous with respect to weak convergence. Thus $T\nu_{n_j} \to T\mu$ weakly. Therefore

$$T^{n_j}\nu/n_j - \nu/n_j = T\nu_{n_j} - \nu_{n_j} \to T\mu - \mu$$

weakly as $j \to \infty$. But the left side converges weakly to 0, so $T\mu = \mu$, i.e. μ is stationary.

Suppose now that μ is any stationary probability distribution and let $\lambda > 0$ and $M_1(\lambda) < 1$ or $\lambda < 0$ and $M_0(\lambda) < 1$. For any $d > 0$ let $F(L) = \min(e^{\lambda L}, d)$. Then

$$\int_{-\infty}^{\infty} F(L)\, \mu(dL) = \int_{-\infty}^{\infty} F(L)\, T^n\mu(dL) = \int_{-\infty}^{\infty} U^nF(L)\, \mu(dL).$$

Since $U^nF(L) \leqslant d$ for all n and L, Fatou's lemma gives

$$\int_{-\infty}^{\infty} F(L)\, \mu(dL) \leqslant \int_{-\infty}^{\infty} \limsup U^nF(L)\, \mu(dL),$$

so $\int_{-\infty}^{\infty} F(L)\, \mu(dL) \leqslant B(\lambda)$ by Lemma 1. As $d \to \infty$ the left hand side converges to $\int_{-\infty}^{\infty} e^{\lambda L}\mu(dL)$, so $\int_{-\infty}^{\infty} e^{\lambda L}\mu(dL) \leqslant B(\lambda)$. Q.E.D.

The continuity of T insures that S is weakly closed, and the inequality

$$\int_{-\infty}^{\infty} \cosh \lambda L \mu(dL) \leqslant (B(\lambda) + B(-\lambda))/2,$$

valid for all $\mu \in S$ if λ is sufficiently small, implies that S is weakly compact. Therefore, by the Krein-Milman theorem (Dunford and Schwartz, 1958, Theorem V.8.4), the convex set S is the closed convex hull of its extremal points. These extremal points are precisely the ergodic probability measures (Rosenblatt, 1967, Theorem 2).

The stationary probability distributions represent the possible modes of steady state behavior of $\{L_n\}$. At the present time we are in the unfortunate position of being unable to prove that such behavior develops asymptotically for all initial states. Let $\nu_{n,L}(A) = (1/n) \sum_{j=1}^{n} P_L(L_j \in A)$. One would like to know what conditions, if any, must be added to the hypotheses of Theorem 5 to insure that $\nu_{n,L}$ converges weakly for every L. We saw in the proof of Theorem 5 that the sequence $\nu_{n,L}$ is conditionally weakly compact and that any subsequential limit is stationary. It follows that a sufficient, though by no means necessary, condition for convergence of $\nu_{n,L}$ is that there be a unique stationary probability distribution ν, in which case $\nu_{n,L} \to \nu$ as $n \to \infty$ for all L. The question of how to supplement the hypotheses of Theorem 5 to insure that S is a unit set is in need of investigation. If Q_i is concentrated at a point a_i (thus $a_i = m_i$) and a_0/a_1 is rational, say j/k, then $L_n - L$ is concentrated on the set Y of integer multiples of $\epsilon = a_0/j = a_1/k$. The proof of the first part of Theorem 5 shows that there exists $\mu_L \in S$ concentrated on $Y + L$. Since $Y + L$ and $Y + L'$ are disjoint unless $L - L' \in Y$, there is more than one stationary probability distribution. J. Pickands and I have proved uniqueness for irrational a_0/a_1 and increasing p.

6. Learning by Small Steps

In this section we assume that the increment ΔL_n in response strength is proportional to a learning rate parameter θ, and study $\{L_n\}$ when θ is small. More precisely, suppose that Q_0 and Q_1 satisfy A. Then $Q_i^\theta(D) = Q_i(D/\theta)$ does too, for any $0 < \theta < \delta$. (If Q_i has mass π_{ij} at b_{ij} as in Sec. 1, then Q_i^θ has mass π_{ij} at θb_{ij}.) Suppose, in addition to B, that p has two bounded derivatives, and that $p'(L) > 0$ for all $L \in R$ (e.g. $p = v/(v+1)$). Suppose, finally, that $\{(L_n^\theta, X_n^\theta)\}$ satisfies C with respect to Q_i^θ. If $L_1^\theta = L$ a.s. for all θ, then there are functions $f(t, L)$ and $g(t, L)$ such that L_{n+1}^θ is approximately normally distributed with mean $f(n\theta, L)$ and variance $\theta g(n\theta, L)$ when θ is small and $n\theta$ is bounded (Norman, 1968, Lemma 2.2.3(i) and Theorem 2.3). The latter restriction limits the approximation to the transient phase of learning. Theorem 6 gives analogous information about steady state behavior in the recurrent case. This result is an extension of the central limit theorem of Norman and Graham (1968) to an unbounded state space.

Note that $m_i^\theta = \theta m_i$, so that $m_0 > 0$ and $m_1 < 0$ implies $m_0^\theta > 0$ and $m_1^\theta < 0$. In this case, let S_θ be the set of stationary probability distributions corresponding to θ. Let $\Lambda = p^{-1}(\rho)$, $m_{i,k} = \int_{-\infty}^{\infty} x^k Q_i(dx)$, and

$$\sigma^2 = [m_{1,2}\rho + m_{0,2}(1-\rho)]/2(m_0 - m_1)\,p'(\Lambda).$$

Theorem 6. *Suppose $m_0 > 0$ and $m_1 < 0$. For every $0 < \theta < \delta$ let $\mu_\theta \in S_\theta$. Then μ_θ is asymptotically normal with mean Λ and variance $\theta\sigma^2$ as $\theta \to 0$.*

Proof. Suppose that L_1^θ has the distribution μ_θ. Then the same is true of all L_n^θ. We first show that $E[(L_n^\theta - \Lambda)^2] = O(\theta)$. Henceforth, we drop θ superscripts.

$$\begin{aligned} E[(L_{n+1} - \Lambda)^2] &= E[((L_n - \Lambda) + \Delta L_n)^2] \\ &= E[(L_n - \Lambda)^2] + 2\theta E[(L_n - \Lambda)\, w_1(L_n)] + \theta^2 E[w_2(L_n)], \end{aligned}$$

where

$$w_k(L) = E[(\Delta L_n/\theta)^k \mid L_n = L] = m_{1,k}p + m_{0,k}q.$$

Thus, cancelling the term on the left and the first term on the right, and noting that $|w_k(L)| \leqslant \max(|m_{1,k}|, |m_{0,k}|) = a_k$, we obtain $E[(\Lambda - L_n)\, w_1(L_n)] \leqslant \theta a_2/2$. Now $w_1(L) = (m_0 - m_1)(\rho - p)$ is a strictly decreasing function with $w_1(\Lambda) = 0$, and $w_1'(\Lambda) = -(m_0 - m_1)\,p'(\Lambda) < 0$. Thus, $(\Lambda - L)\, w_1(L) \geqslant 0$, and $w_1(L)/(\Lambda - L)$ is bounded away from 0 if $|\Lambda - L|$ is bounded. It follows that, for any $d > 0$,

$$E[(L_n - \Lambda)^2 I_{|L_n - \Lambda| \leqslant d}] = O(\theta). \tag{4}$$

Expanding $E[((L_n - \Lambda) + \Delta L_n)^4]$ we get

$$E[(\Lambda - L_n)^3 w_1(L_n)] \leqslant (3/2)\theta a_2 E[(L_n - \Lambda)^2] + \theta^2 a_3 E[|L_n - \Lambda|] + \theta^3 a_4/4.$$

Now $\epsilon = \inf_{|L-\Lambda|>d}(\Lambda - L)\, w_1(L) > 0$, so the left side is at least $\epsilon\alpha$, where $\alpha = E[(L_n - \Lambda)^2 I_{|L_n-\Lambda|>d}]$. Furthermore, $E[(L_n - \Lambda)^2] \leqslant \alpha + O(\theta)$ by (4), and $E[|L_n - \Lambda|] \leqslant d + d^{-1}\alpha$. It follows that $\alpha = O(\theta^2)$. Combining this with (4) we obtain $E[(L_n - \Lambda)^2] = O(\theta)$.

The remainder of the proof of the central limit theorem of Norman and Graham (1968) goes through without essential modification. The variable p_n there corresponds to L_n here, and it is unnecessary to let $n \to \infty$ since L_n has the distribution μ_θ for all n. Q.E.D.

Theorem 6 implies that the stationary probability distributions of $\{p_n{}^\theta\}$ are asymptotically normal with mean ρ and variance θs^2, where

$$s^2 = p'(\Lambda)[m_{1,2}\rho + m_{0,2}(1 - \rho)]/2(m_0 - m_1),$$

as $\theta \to 0$.

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