Sequential Processes and the Shapes of Reaction-Time Distributions

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It is sometimes suggested that reaction-time (RT) distributions have the same shape across conditions or groups. In this note I show that this is highly unlikely if the RT is the sum of the stochastically independent durations of two or more stages (sequential processes) (a) that are influenced selectively by different factors, or (b) one of which is influenced selectively by some factor.

It has sometimes been suggested that reaction-time (RT) distributions have the same shape across conditions or groups. (Shape invariance of a set of distributions means that they differ by at most their means and time scales.[2]) For example, one proposal about the cognitive effects of aging is the General Slowing Hypothesis: With increasing age, all the operations of the central nervous system in most or all tasks become proportionally slower (Cerella, 1985; Eckert, 2011; Myerson et al., 2003a; Myerson et al., 2003b; Salthouse, 1996; Sleiman-Malkoun et al., 2013). In effect, with increasing age, time runs more slowly. Rouder et al. (2010) discuss other considerations that lead to shape invariance. Ratcliff and McKoon (2008) use the approximate linearity of Q-Q plots to show that for diffusion model predictions and some data sets, the shapes of RT distributions are approximately invariant across experimental conditions and experiments (p. 895). According to Ratcliff and Smith (2010, p. 90), "Invariance of distribution shape is one of the most powerful constraints on models of RT distributions....That the diffusion model predicts this invariance is a strong argument in support of its use in performing process decomposition of RT data."

The purpose of this note is to show that for a process organized in stages that have stochastically independent durations and are selectively influenced by experimental factors, it is highly unlikely that the distributions of RTs in several conditions in an experiment can have the same shape.[3]

Stage models are ubiquitous in the reaction-time literature (e.g., Sanders, 1998; Sternberg, 1998). For several sets of data, Roberts & Sternberg (1993) provide evidence for selectively influenced stages whose durations are stochastically independent. Even in Ratcliff's (1978) diffusion model, in which the decision process may be represented by multiple activations that grow in parallel, the decision stage is augmented by two additional processes arranged sequentially, an initial stage for stimulus encoding, and a final stage for response execution. In the application of the diffusion model considered by Gomez, Perea, & Ratcliff (2013), the duration of the encoding stage in a lexical-decision task is found to be selectively influenced by the relatedness of masked primes. In the experiments considered by Ratcliff & Smith (2010), an

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2. The distributions of X and Y have the same shape iff there are constants a and b such that $Y = a + bX$.

3. That stage models with stochastically independent durations imply constraints on RT distributions surprised me. In another report (Sternberg, 2014) I show that such models are inconsistent with the popular ex-Gaussian distribution.

initial encoding stage is inferred that delays the start of the second stage ("decision making") by an amount that is changed considerably (by 100 ms or more) by variations in stimulus noise; because the same factor also affects the decision stage, however, its influence with respect to those two stages is not selective.[4]

Two stages with selective effects on both. Because two or more stages can be concatenated and treated as a single stage, we can limit consideration to processes consisting of two stages without loss of generality. Consider a process that consists of two stages **A** and **B**, with durations T_A and T_B, so that the reaction time is $RT = T_A + T_B$. Assume that T_A and T_B are stochastically independent. We then have an *SIStage process* (a process consisting of stages whose durations are stochastically independent; Roberts & Sternberg, 1993). Suppose two factors, F_j and G_k, each with two levels, $j = 1, 2$, and $k = 1, 2$, that influence the stage durations selectively, so that $T_A = T_A(F_j) = T_{Aj}$, $T_B = T_B(G_k) = T_{Bk}$, and $RT_{jk} = T_{Aj} + T_{Bk}$. Consider a 2×2 factorial experiment with the four resulting conditions, giving us RT_{11}, RT_{12}, RT_{21}, and RT_{22}. (Because an $m \times n$ experiment can be regarded as a concatenation of 2×2 experiments, we can do so without loss of generality.) Then, because convolution is associative and commutative, $RT_{11} + RT_{22}$ has the same distribution as $RT_{12} + RT_{21}$, namely, the convolution of the distributions of T_{A1}, T_{A2}, T_{B1}, and T_{B2}. Thus,

$$RT_{11} * RT_{22} = RT_{12} * RT_{21} \quad , \tag{1}$$

where * represents convolution[5] , and therefore:

$$\kappa_{r11} + \kappa_{r22} = \kappa_{r12} + \kappa_{r21} \quad , \quad (r \geq 1), \tag{2}$$

where κ_{rjk} is the r^{th} cumulant of RT_{jk} . Let us assume that RT distributions are "well-behaved", in the sense that cumulants of (at least) orders $r = 1, 2, 3$, and 4 exist.[6]

Eq. 2 results from three assumptions: (a) *Stages*, (b) *Stochastic Independence*, and (c) *Selective Influence*. To these, let us add a fourth assumption: (d) *Shape Invariance*: The RT_{jk} distributions differ by at most means and scale factors. Whereas differences among means influence only the means of the RT_{jk} distributions and influence none of the cumulants above the first, scale factor differences influence all of the cumulants and central moments above the first. Let the four scale factors be C_{11}, C_{12}, C_{21}, and C_{22}, with $C_{jk} > 0$, and note that:

(3)

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4. It appears that the durations of the three stages in the diffusion model are assumed to be stochastically independent, but, surprisingly, the sum of the durations of encoding and response stages, a sum whose mean is estimated to be as much as 600 ms (Ratcliff & Smith, 2010, p. 89), is assumed to have a rectangular distribution. This assumption cannot be correct if neither of these two distributions is degenerate, and if the two components are unimodal and don't have opposite skewness.

5. This is the basis of the *summation test*, introduced with applications by Roberts & Sternberg (1993). See their Section 26.2 and Appendix for a more formal proof and a discussion. See also Ashby and Townsend, 1980.

6. In what follows, two well-known properties of cumulants (κ_r) of order r are used: For stochastically independent random variables X and Y, $\kappa_r(X+Y) = \kappa_r(X * Y) = \kappa_r(X) + \kappa_r(Y)$ and $\kappa_r(CX) = C^r \kappa_r(X)$. Note that $\kappa_{rjk} = M_{rjk}$, for $1 \leq r \leq 3$ and $\kappa_{4jk} = M_{4jk} - 3M_{2jk}^2$, where the $\{M_{rjk}\}$ are the mean and r^{th} central moments of RT_{jk}. (See Kendall & Stuart, Volume 1, 1969, Ch. 3.) The quantity κ_4/κ_2^2 is a common measure of kurtosis, whose value is zero for the Gaussian distribution, and nonzero for other common distributions (Weisstein, 2014).

$$\kappa_{rjk} = C^r_{jk}\kappa_{r00} \quad , \quad (r \geq 2, \;\; \kappa_{r00} \neq 0).$$

where the $\{\kappa_{r00}\}$ are a set of constants, one for each r. With Assumption (d), Eqs. 2 and 3 then imply that:

$$C^r_{11} + C^r_{22} = C^r_{12} + C^r_{21} \;, \quad (r \geq 2, \;\; \kappa_{r00} \neq 0). \tag{4}$$

Given that $\kappa_{200} \neq 0$ (nonzero variances) and $\kappa_{400} \neq 0$ (nonzero kurtosis values, which, among common distributions, excludes only the Gaussian) there are only three relations among the C_{jk} that satisfy Eq. (4)[7]:
(α) the C_{jk} are identical, or (β) $C_{11} = C_{21}$ and $C_{22} = C_{12}$, or (γ) $C_{11} = C_{12}$ and $C_{22} = C_{21}$. Given (α), only the mean RT and none of the higher cumulants can be influenced by either factor. Given (β), factor F can influence only the mean. Given (γ), factor G can influence only the mean. Thus, for the four RT distributions to have the same shape, at least one of the two factors can cause no more than a *shift* (a change in mean only) of the RT distribution, a highly unlikely possibility.[8]

Two stages with a selective effect on one. [9] Suppose the two-stage model, with one factor F_j ($j = 1, 2, , \ldots$) that influences just T_A, so that

$$RT_j = T_{Aj} + T_B \;, \quad (j \geq 1). \tag{5}$$

We then have

$$\kappa_{rj} = \alpha_{rj} + \beta_r \;, \quad (j \geq 1, \, r \geq 1), \tag{6}$$

where κ_{rj}, α_{rj}, and β_r are the r^{th} cumulants of RT_j, T_{Aj}, and T_B, respectively. Equation (6) follows from assumptions (a), (b), and (c), above. Addition of the *Shape Invariance* assumption then requires

$$\kappa_{rj} = C^r_j\kappa_{r1} \quad , \quad (j \geq 2, \, r \geq 2), \tag{7}$$

where $C_j \neq 1$ is the scale factor that relates RT_j to RT_1. Combining Eqs. (6) and (7) and rearranging, we have

$$\beta_r = \frac{(\alpha_{rj} - C^r_j\alpha_{r1})}{(C^r_j - 1)} \quad , \quad (j \geq 2 \;, \, r \geq 2). \tag{8}$$

Thus, either T_B is a constant ($\beta_r = 0, \;\; r \geq 2$) or its distribution (which is uniquely determined up

7. To prove this, use Eq. (4) with $r = 2$ and $r = 4$. For simplicity, let $C_{11} = a$, $C_{22} = b$, $C_{12} = c$, and $C_{21} = d$. Start with (5) $a^2 + b^2 = c^2 + d^2$ and (6) $a^4 + b^4 = c^4 + d^4$. Express the two sides of (5) and (6), respectively, in terms of $(a + b)^2$ and $(c + d)^2$, and of $(a + b)^4$ and $(c + d)^4$. Square the two sides of the equation derived from (5), and subtract from the equation derived from (6). This gives $ab = cd$, or $a/c = d/b = k$. Substituting in (5) gives $(k^2 - 1)(c^2 - b^2) = 0$. This implies either that $k = 1$, which means that $a = c$ and $b = d$; or that $b = c$, which requires $a = d$.

8. Effects of factors on mean RT are almost always associated with non-zero effects on other aspects of the distribution, including var(RT). Indeed, Wagenmakers & Brown (2007) have argued for a lawful regularity in the relation between mean and variance: they claim that the standard deviations of RT distributions increase linearly with their means.

9. This is sometimes assumed or concluded in applications of Ratcliff's (1978) diffusion model.

to its mean by the $\{\beta_r\}$, $r \geq 2$) is restricted by properties of the distribution of T_A, and may vary with the level of the factor F_j that is assumed to influence only T_A, a contradiction.

We can draw two conclusions: First, given stages with variable and independent durations, and factors that influence more than the means of those durations selectively, the RT distributions in a factorial experiment are highly unlikely to have the same shape. Second, given variable and independent durations, and a factor that influences more than the mean of just one of those durations, the RT distributions for different levels of that factor are highly unlikely to have the same shape. How well the distributions produced by an SIStage process can *approximate* shape invariance is a question for further research. In thinking about this issue it is important to consider the sensitivity of the standard tests for differences between distributions, and the associated graphical displays.

REFERENCES

Ashby, F. G. & Townsend, J. T. (1980). Decomposing the reaction-time distribution: Pure insertion and selective influence revisited. *Journal of Mathematical Psychology, 21,* 93-123.

Cerella, J. (1985). Information processing rates in the elderly. *Psychological Bulletin, 98* 67-83.

Eckert, M. A. (2011). Slowing down: Age-related neurobiological predictors of processing speed. *Frontiers in Neuroscience 5:25,* doi: 10:3389/fnins.2011.00025.

Gomez, P., Perea, M., & Ratcliff, R. (2013). A diffusion model account of masked versus unmasked priming: Are they qualitatively different? *Journal of Experimental Psychology: Human Perception and Performance, 39,* 1731-1740.

Kendall, M. G. & Stuart, A. (1969). *The advanced theory of statistics, Volume 1.* New York: Hafner.

Myerson, J., Adams, D. R., Hale, S., & Jenkins, L. (2003a). Analysis of group differences in processing speed: Brinley plots, Q-Q plots, and other conspiracies. *Psychonomic Bulletin & Review, 10,* 224-237.

Myerson, J., Hale, S., Zheng, Y., Jenkins, L., & Widaman, K. W. (2003b). The difference engine: A model of diversity in speeded cognition. *Psychonomic Bulletin and Review, 10,* 262-288

Ratcliff, R. (1978). A theory of memory retrieval. *Psychological Review, 85,* 59-108.

Ratcliff, R. & McKoon, G. (2008). The diffusion decision model: Theory and data for two-choice decision tasks. *Neural Computation, 20,* 873-922.

Ratcliff, R. & Smith, P. L. (2010). Perceptual discrimination in static and dynamic noise: The temporal relation between perceptual encoding and decision making. *Journal of Experimental Psychology: General, 139,* 70-94.

Roberts, S. & Sternberg, S. (1993). The meaning of additive reaction-time effects: Tests of three alternatives. In D. E. Meyer & S. Kornblum (Eds.), *Attention and performance XIV: Synergies in experimental psychology, artificial intelligence, and cognitive Neuroscience* (pp. 611-653). Cambridge, MA : M.I.T. Press.

Rouder, J. N., Yue, Y., Speckman, P. L., Pratte, M. S., & Province, J. M. (2010). Gradual growth versus shape invariance in perceptual decision making. *Psychological Review, 117,* 1267-1274.

Salthouse, T. A. (1996). The processing-speed theory of adult age differences in cognition. *Psychological Review, 103,* 403-428.

Sanders, A. F. (1998). *Elements of human performance: Reaction processes and attention in human skill.* Mahwah, NJ: Erlbaum.

Sleimen-Malkoun, R. Temprado, J.-J., & Berton. E. (2013). Age-related dedifferentiation of cognitive and motor slowing: Insight from the comparison of Hick-Hyman and Fitts' laws. *Frontiers in Aging Neuroscience 5:62,* doi: 10.3389/fnagi.2013.00062.

Sternberg, S. (1998). Discovering mental processing stages: The method of additive factors. In D. Scarborough & S. Sternberg (Eds.) *An invitation to cognitive science, Volume 4: Methods, models, and conceptual issues.* (pp. 703-863). Cambridge, MA: M.I.T. Press.

Sternberg, S. (2014). *Reaction times and the ex-Gaussian distribution: When is it appropriate?* Retrieved from http://www.psych.upenn.edu/~saul/exgauss.stages.pdf

Wagenmakers, E.-J. & Brown, S. (2007) On the linear relation between the mean and the standard deviation of a response time distribution. *Psychological Review, 114,* 830-841.

Weisstein, E. W. (2014) Kurtosis. *MathWorld--A Wolfram Web Resource.* Retrieved from http://mathworld.wolfram.com.